Eigenvalues & Gen. Eigenvectors: One Last Time.

To develop our intuition, let us start with a 6×6 matrix A given by

$$A = \left(\begin{array}{ccc|cc|c} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda \end{array}\right)$$

A has eigenvalue λ repeated 6 times.

Calculate

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$A - \lambda I$ has a 3 dimensional null space spanned by the vectors

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{= e_1}, \quad \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{= e_4}, \quad \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{= e_6}$$

e_1, e_4, e_6 are eigenvectors of A that are l.i.

Remark! Only 3 l.i. eigenvectors.

The other vectors would be gen. eigenvectors. To find them, let us proceed as follows.

Calculate

$$(A-\lambda I)^2 =$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Null space of $(A-\lambda I)^2$ is spanned by

$$\{e_1, e_2, e_4, e_5, e_6\}$$

Calculate

$(A-\lambda I)^3 = 0$ matrix

Null space of $(A-\lambda I)^3$ is spanned by

$\{e_1, e_2, e_3, e_4, e_5, e_6\}$.

Q: Find l.i. non-zero vectors which are in the null space of $(A-\lambda I)^3$ but not in the null space of $(A-\lambda I)^2$.

Ans: Such a vector is e_3 and all vectors that are scalar multiples of e_3.

we define a chain of gen. eigenvectors

$$u_1 = e_3$$
$$u_2 = (A-\lambda I)e_3 = e_2$$
$$u_3 = (A-\lambda I)^2 e_3 = e_1$$

$$u_1 \downarrow u_2 \downarrow u_3$$

Q: Find l.i. vectors in the null space of $(A-\lambda I)^2$ but not in the null space of $(A-\lambda I)$. These vectors are additional l.i. with respect to u_1, u_2, u_3 already picked.

Ans: e_2 and e_5 are l.i. vectors in the null space of $(A-\lambda I)^2$ and not in the null space of $(A-\lambda I)$.

However e_2 has to be deleted because it is not l.i. with respect to u_1, u_2, u_3 ($u_2 = e_2$).

Only admissible vector is e_5 and we define

$$u_4 = e_5$$

$$u_5 = (A-\lambda I)e_5 = e_4.$$

$$u_4 \to u_5$$

Q. Find l.i. vectors in the nullspace of $(A-\lambda I)$.
These vectors are additionally l.i. with respect to u_1, u_2, u_3, u_4, u_5 already picked.

Ans: e_1, e_4, e_6 are vectors in the null space of $A-\lambda I$. e_1 & e_4 have to be deleted because they have already been picked. e_6 is the only remaining vector and we define

$$u_6 = e_6.$$

In this problem, we have 3 chains.